Systems of Particles and Rotational Motion

Class 11 Notes Physics Chapter 7

A rigid body is a body with a perfectly definite and unchanging shape. The distances between all pairs of particles of such a body do not change.

• Centre of Mass

For a system of particles, the centre of mass is defined as that point where the entire mass of the system is imagined to be concentrated, for consideration of its translational motion.

If all the external forces acting on the body/system of bodies were to be applied at the centre of mass, the state of rest/ motion of the body/system of bodies shall remain unaffected.

• The centre of mass of a body or a system is its balancing point. The centre of mass of a two- particle system always lies on the line joining the two particles and is somewhere in between the particles.

If there are two particles of masses m_1 and m_2 having position vectors $\vec{r}_1$ and $\vec{r}_2$, then the position vector of the centre of mass is given by

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Special Note: If the masses are of equal magnitude the centre of mass lies at the mid-point of the line joining them. If the masses are unequal, centre of mass is closer to the heavier body.

- For a system of n particles of masses m_1, m_2, m_3 .... m_n and their respective position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$ .... $\vec{r}_n$; the position

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots\dots\dots + m_n\vec{r}_n}{m_1 + m_2 + \dots\dots\dots m_n} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

- The co-ordinates of the centre of mass of an n-particle system is given as:

$$X = \frac{m_1x_1 + m_2x_2 + \dots\dots\dots m_nx_n}{m_1 + m_2 + \dots\dots\dots m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_ix_i$$

where $\sum_{i=1}^{n} m_i = M$, mass of system.

$$Y = \frac{m_1 y_1 + m_2 y_2 + \dots\dots\dots\dots m_n y_n}{m_1 + m_2 + \dots\dots\dots\dots m_n} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_{i=1}^{n} m_i y_i$$

$$Z = \frac{m_1 z_1 + m_2 z_2 + \dots\dots\dots\dots m_n z_n}{m_1 + m_2 + \dots\dots\dots\dots m_n} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M} \sum_{i=1}^{n} m_i z_i$$

• Motion of centre of Mass

The centre of mass of a system of particles moves as if the entire mass of the system were concentrated at the centre of mass and all the external forces were applied at that point. Velocity of centre of mass of a system of two particles, m_1 and m_2 with velocity v_1 and v_2 is given by,

$$V_{cm} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

Acceleration of centre of mass, a_{cm} of a two body system is given by

$$a_{cm} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$

• If no external force acts on the body, then the centre of mass will have constant momentum. Its velocity is constant and acceleration is zero,

i.e., MV_{cm} = constant.

• Vector Product or Cross Product of two vectors

The vector product or cross product of two vectors $\vec{A}$ and $\vec{B}$ is another vector $\vec{C}$, whose magnitude is equal to the product of the magnitudes of the two vectors and sine of the smaller angle between them.

If θ is the smaller angle between $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = \vec{C} = AB \sin\theta \hat{C}$$

where $\hat{C}$ is a unit vector in the direction of $\vec{C}$. The direction of $\vec{C}$ or $\hat{C}$ (*i.e.,* vector product of two vectors) is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and pointing in the direction of advance of a right handed screw when rotated from $\vec{A}$ to $\vec{B}$.

- Some important properties of cross-product are as follows:
  - (*a*) For parallel as well as anti-parallel vectors (*i.e.,* when $\theta = 0°$ or $180°$), the cross-product is zero.
  - (*b*) The magnitude of cross-product of two perpendicular vectors is equal to the product of the magnitudes of the given vectors.
  - (*c*) Vector product is anti-commutative *i.e.,* $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
  - (*d*) Vector product is distributive *i.e.,* $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
  - (*e*) $\vec{A} \times \vec{B}$ does not change sign under reflection *i.e.,* $(-\vec{A}) \times (-\vec{B}) = \vec{A} \times \vec{B}$

(*f*) For unit orthogonal vectors, we have

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0, \quad \hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i} \quad \text{and} \quad \hat{k}\times\hat{i} = \hat{j}$$

Moreover $\hat{j}\times\hat{i} = -\hat{k}, \quad \hat{k}\times\hat{j} = -\hat{i} \quad \text{and} \quad \hat{i}\times\hat{k} = -\hat{j}$

(*g*) In terms of components $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$.

- The angular velocity of a body or a particle is defined as the ratio of the angular displacement of the body or the particle to the time interval during which this displacement occurs.

$$\omega = \frac{d\theta}{dt}$$

The direction of angular velocity is along the axis of rotation. It is measured in radian/sec and its dimensional formula is $[M^0L^0T^{-1}]$.

The relation between angular velocity and linear velocity is given by

$$\boxed{\vec{v} = \vec{\omega}\times\vec{r}}$$

- The angular acceleration of a body is defined as the ratio of the change in the angular velocity to the time interval.

$$\text{Angular acceleration} = \frac{\text{Change in angular velocity}}{\text{time taken}}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

The unit of angular acceleration is rad s^{-2} and dimensional formula is $[M^0L^0L^{-2}]$.

• Torque

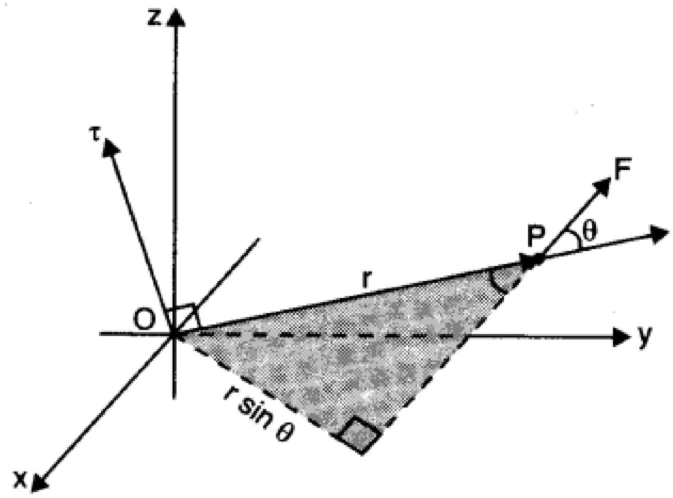
Torque is the moment of force. Torque acting on a particle is defined as the product of the magnitude of the force acting on the particle and the perpendicular distance of the application of force from the axis of rotation of the particle.

Torque or moment of force = force × perpendicular distance

$$\vec{\tau} = \vec{r}\times\vec{F} = rF\sin\theta\,\hat{n}$$

where θ is smaller angle between $\vec{r}$ and $\vec{F}$; $\hat{n}$ is unit vector along $\vec{r}$.

It is measured in Nm and has dimensions of $[ML^2T^{-2}]$.



• Angular Momentum

The angular momentum (or moment of momentum) about an axis of rotation is a vector quantity, whose magnitude is equal to the product of the magnitude of momentum and the perpendicular distance of the line of action of momentum from the axis of rotation and its direction is perpendicular to the plane containing the momentum and the perpendicular distance.

It is given by

$$\vec{L} = \vec{r} \times \vec{p}$$

SI unit of angular momentum is kg m^2s^{-1} and its dimensional formula is $[M^1L^2T^{-1}]$.

• Geometrically, the angular momentum of a particle is equal to twice the product of its mass and the areal velocity, *i.e.,*

$$L = 2\text{ m} \times \frac{dA}{dt}$$

• Torque (τ) and angular momentum are corelated as:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

• If no net external torque acts on a system then the total angular momentum of the system remains conserved. Mathematically, if $\vec{\tau}_{ext} = \vec{0}$, then

$$\vec{L} = \vec{L}_1 + \vec{L}_2 + \ldots\ldots + \vec{L}_n = \text{a constant}$$

• Axis of Rotation

A rigid body is said to be rotating if every point mass that makes it up, describes a circular path of a different radius but the same angular speed. The circular paths of all the point masses have a common centre. A line passing through this common centre is the axis of rotation.

• A rigid body is said to be in equilibrium if under the action of forces/torques, the

body remains in its position of rest or of uniform motion.
For translational equilibrium, the vector sum of all the forces acting on a body must be zero. For rotational equilibrium, the vector sum of torques of all the forces acting on that body about the reference point must be zero. For complete equilibrium, both these conditions must be fulfilled.

• Couple

Two equal and opposite forces acting on a body but having different lines of action, form a couple. The net force due to a couple is zero, but they exert a torque and produce rotational motion.

• Moment of Inertia

The rotational inertia of a rigid body is referred to as its moment of inertia.
The moment of inertia of a body about an axis is defined as the sum of the products of the masses of the particles constituting the body and the square of their respective perpendicular distance from the axis.
It is given by .

$$I = m_1 r_1^2 + m_2 r_2^2 + \dots\dots\dots\dots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2,$$

where m_i is the mass and r_i the distance of the i^{th} particle of the rigid body from the axis of rotation.
It is measured in kg m^2 and has the dimension of $[ML^2]$.

• Radius of Gyration

The distance of a point in a body from the axis of rotation, at which if whole of the mass of the body were supposed to be concentrated, its moment of inertia about the axis of rotation would be the same as that determined by the actual distribution of mass of the body is called radius of gyration.
If we consider that the whole mass of the body is concentrated at a distance K from the axis of rotation, then moment of inertia I can be expressed as $I = MK^2$
where M is the total mass of the body and K is the radius of gyration. It is given as

$$K = \sqrt{\left(\frac{r_1^2 + r_2^2 + \dots + r_n^2}{n}\right)}$$

• Theorem of Parallel Axes

According to this theorem, the moment of inertia I of a body about any axis is equal to its moment of inertia about a parallel axis through centre of mass, I_{cm}, plus Ma^2 where M is the mass of the body and V is the perpendicular distance between the axes, i.e.,
$I = I_{cm} + Ma^2$

• Theorem of Perpendicular Axes

According to this theorem, the moment of inertia I of the body about a perpendicular axis is equal to the sum of moments of inertia of the body about two axes at right angles to each other in the plane of the body and intersecting at a point where the perpendicular axis passes, i.e.,

$$I = I_x + I_y$$

- A body in rotatory motion possesses rotational kinetic energy given by:

$$\text{Rotational } K.E. = \frac{1}{2} I \omega^2.$$

- In terms of moment of inertia of a body, its angular momentum is defined as the product of moment of inertia and angular velocity *i.e.,*

$$\vec{L} = I \vec{\omega}$$

- Torque may be defined as the produce of moment of inertia and the angular acceleration *i.e.,*

$$\vec{\tau} = I \vec{\alpha}$$

• Rolling Motion

The combination of rotational motion and the translational motion of a rigid body is known as rolling motion.

The kinetic energy associated with a body rolling is the sum of the translational and rotational kinetic energies, *i.e.,* $K.E$ of rolling $= \frac{1}{2} Mv^2 + \frac{1}{2} I \omega^2$

- When a body rolls down an inclined plane (θ) without slipping, the velocity on reaching the ground is,

$$v = \sqrt{\frac{2gh}{1 + \frac{K^2}{r^2}}},$$

where h is the vertical height of inclined plane and K is the radius of gyration of the rolling body.

- The acceleration of a body rolling down an inclined plane is given by

$$a = \frac{g \sin \theta}{\left(1 + \frac{K^2}{r^2}\right)}.$$

• Law of Conservation of Angular Momentum

According to the law of conservation of angular momentum, if there is no external couple acting, the total angular momentum of a rigid body or a system of particles is conserved.

If the moment of inertia of the body changes from I_1 to I_2 due to the change of the distribution of mass of the body, then angular velocity of the body changes from $\vec{\omega}_1$ to $\vec{\omega}_2$, such that

$$I_1 \vec{\omega}_1 = I_2 \vec{\omega}_1 \quad \text{or} \quad I_1 \omega_1 = I_2 \omega_2.$$

IMPORTANT TABLES

TABLE 7.1

Quantity	*Symbol*	*Dimensions*	*Units*	*Remarks*
Angular velocity	ω	$[T^{-1}]$	rad s^{-1}	$\vec{v} = \vec{\omega} \times \vec{\gamma}$
Angular Momentum	L	$[ML^2T^{-1}]$	J s	$\vec{L} = \vec{r} \times \vec{p}$
Torque	τ	$[ML^2T^{-2}]$	N m	$\vec{\tau} = \vec{r} \times \vec{F}$
Moment of inertia	I	$[ML^2]$	kg m^2	$I = \sum m_i r_i^2$

TABLE 7.2 Analogy between linear motion and rotational motion

	Linear Motion		Rotational Motion
1.	Distance/displacement (s)	**1.**	Angle or angular displacement (θ)
2.	Linear velocity $v = \frac{ds}{dt}$	**2.**	Angular velocity $\omega = \frac{d\theta}{dt}$
3.	Linear acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$	**3.**	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
4.	Mass (m)	**4.**	Moment of inertia (I)
5.	Linear momentum $P = mv$	**5.**	Angular momentum $L = I\omega$
6.	Force $F = ma$	**6.**	Torque $\tau = I\,\alpha$
7.	Also, force $F = \frac{dp}{dt}$	**7.**	Also, torque $\tau = \frac{dL}{dt}$
8.	Translational K.E $= \frac{1}{2}mv^2 = \frac{p^2}{2m}$	**8.**	Rotational K.E $= \frac{1}{2}I\omega^2 = \frac{L^2}{2I}$
9.	Work done, $W = Fs$	**9.**	Work done, $W = \tau\,\theta$
10.	Power $P = Fv$	**10.**	Power $= \tau\,\omega$
11.	Linear momentum of a system is conserved when no external force acts on the system. (Principle of conservation of linear momentum)	**11.**	Angular momentum of a system is conserved when no external torque acts on the system. (Principle of conservation of angular momentum)
12.	Equations of Translational motion (*i*) $v = u + at$ (*ii*) $s = ut + \frac{1}{2}at^2$ (*iii*) $v^2 - u^2 = 2as$, where the symbols have their usual meaning.	**12.**	Equations of Rotational motion (*i*) $\omega_2 = \omega_1 + \alpha t$ (*ii*) $\theta = \omega_1\, t + \frac{1}{2}\alpha t^2$ (*iii*) $\omega_2^2 - \omega_1^2 = 2\,\alpha\,\theta$, where the symbols have their usual meaning.
13.	Distance travelled in nth second $S_{nth} = u + \frac{a}{2}(2n-1)$	**13.**	Angle traced in nth second $\theta_{nth} = \varphi_1 + \frac{\alpha}{2}(2n-1)$

TABLE 7.3 Moment of Inertia of some bodies of regular shape

S. No.	Body	Axis	Moment of Inertia
1.	Uniform rod of length l	perpendicular to rod through its centre	$\frac{1}{12} M l^2$
2.	Uniform rectangular lamina of length l and breadth b	perpendicular to lamina and through its centre	$M\left(\frac{l^2+b^2}{12}\right)$
3.	Uniform circular ring of radius R	perpendicular to its plane and through the centre	MR^2
4.	Uniform circular ring of radius R	Diameter	$MR^2/2$
5.	Uniform circular disc of radius R	perpendicular to its plane and through the centre	$\frac{1}{2}MR^2$
6.	Uniform circular disc of radius R	Diameter	$\frac{1}{4}MR^2$
7.	Hollow cylinder of radius R	Axis of cylinder	MR^2
8.	Solid cylinder of radius R	Axis of cylinder	$\frac{1}{2}MR^2$
9.	Hollow sphere of radius R	Diameter	$\frac{2}{3}MR^2$
10.	Solid sphere of radius R	Diameter	$\frac{2}{5}MR^2$